A New Approach to Common Coupled Fixed Point of Caristi Type Contraction on a Metric Space Endowed with a Graph

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Abstract

In this paper we introduced a new notation $G - fg$ – contraction of Caristi type and a new edge preserving property. With help of these we proved a some coupled fixed point results for four maps endowed with a graph in a complete metric space. Also we gave an application to integral equations.

Keywords: Metric spaces with a graph, edge preserving property, coupled fixed point.

1. Introduction

Banach introduced the concept of fixed point theory in 1922([10]), it has been extended and generalized by several authors. Caristi type fixed point theorem is one of these generalizations. It is a modified $E$ - variation principle of Ekeland([12]). In 1976, Caristi proved the following famous fixed point theorem.

**Theorem 1.1**

[11] Let $(X, d)$ be complete metric space and $f: X \to X$ be lower semi continuous function and bounded below function. A mapping $T: X \to X$ is said to be Caristi type map on $X$ dominated by $f$ if $T$ satisfies $d(x, Tx) \leq f(x) - f(Tx)$ for each $x \in X$. Then $T$ has a fixed point.

It is well-known that the Caristi’s fixed point theorem is one of the most valuable generalization of the Banach contraction principle. The concepts of fixed point theory and graph theory were combined by Espinola and Kirk ([4]). Jachymski([5]) came up with an interesting idea of using the language of graph theory in the study of fixed point results.

A graph is an ordered pair $G = (V, E)$, where $V$ is a non-empty set and the elements in $V$ are called vertices or nodes and $E$ is a binary relation on $V$. i.e., $E \subseteq (V \times V)$. The elements of $E$ are called edges.

In this paper we concentrate on directed graphs.

Let $G^{-1}$ be the conversion of the graph $G$, i.e., the graph obtained from $G$ by reversing the direction of edges.

Simply, $E(G^{-1}) = \{(y, x) : (x, y) \in E(G)\}$. A directed graph $G$ is called a oriented graph if $E$ satisfies $E(G^{-1}) = \{(y, x) : (x, y) \in E(G)\}$. Bhaskar and Lakshmikantham ([13]) introduced the coupled fixed points concept, later several authors proved some coupled fixed point theorems in partial metric spaces (see [8, 9]).

In the same line, our aim is to extend some coupled coincidence and coupled common fixed point theorems for nonlinear contractions complete metric spaces endowed with a directed graph. Our results would bring about a more unified approach to the presentation of coupled coincidence and coupled common fixed point theorems for four maps. Also, as an application of our results, we aim to prove a theorem which can be used to test the existence of a solution for some particular integral equations.

**Definition 1.2**

[2] Let $F: X \times X \to X$. If $F(\mu, \nu) = \mu$ and $F(\nu, \mu) = \nu$, for $(\mu, \nu) \in X \times X$ then $(\mu, \nu)$ is said to be a coupled fixed point of $F$.

**Definition 1.3**

[1] An element $(\mu, \nu) \in X \times X$ is called

1. a coupled coincident point (CCP) of mappings $F: X \times X \to X$ and $f: X \to X$ if $f(x) = F(\mu, \nu)$ and $f(y) = F(\nu, \mu)$.

2. a common coupled fixed point(CCFP) of mappings $F: X \times X \to X$ and $f: X \to X$ if $f(x) = F(\mu, \nu)$ and $f(y) = F(\nu, \mu)$.

**Definition 1.4**

[7] Two functions $F: X \times X \to X$ and $f: X \to X$ are said to be commutative on a non empty set $X$ if $f(F(\mu, \nu)) = F(f(\mu), f(\nu))$.

**Definition 1.5**

[5] A function $f: X \to X$ is said to be $G$ – continuous if

(a) $(x_n) \to p$ and $(x_n, x_{n+1}) \in E(G)$ implies $f(x_n) \to f(p)$, where $(x_n)_{n\in\mathbb{N}}$ be a sequence of positive integers.

(b) $(y_n) \to q$ and $(y_n, y_{n+1}) \in E(G^{-1})$ impels $f(y_n) \to f(q)$, where $(y_n)_{n\in\mathbb{N}}$ be a sequence of positive integers.

**Definition 1.6**

[3] A function $S: X \times X \to X$ is said to be $G$ – continuous if

$(x_n) \to p$, $(y_n) \to q$ and $(x_n, y_{n+1}) \in E(G)$, $(y_{n+1})_{n\in\mathbb{N}} \in E(G^{-1})$ implies $S(x_{n+1}, y_{n+1}) \to S(p, q)$ and $S(y_{n+1})_{n\in\mathbb{N}} \to S(q, p)$ as $n \to \infty$, where $(x_n, y_n) \in X \times X$ and $(n_i)_{i\in\mathbb{N}}$ be a sequence of positive integers.
Definition 1.7
[3] Let \((X,d)\) be a complete metric space endowed with a directed graph \(G\). Then the triplet \((X,d,G)\) has property (A) if
(i) for any sequence \(x_n\), \(n \in \mathbb{N}\) in \(X\) such that \(x_n \to p\) and \((x_n, x_{n+1}) \in \mathcal{E}(G)\) implies \((x_n, p) \in \mathcal{E}(G)\)
(ii) for any sequence \(y_n\), \(n \in \mathbb{N}\) in \(X\) such that \(y_n \to q\) and \((y_n, y_{n+1}) \in \mathcal{E}(G^-)\) implies \((y_n, q) \in \mathcal{E}(G^-)\)

Lemma 1.8 [6] Let \(F: M \to M\) be a reflexive relation on a nonempty set \(M\) and \(f: M \to \mathbb{R}\) be a function bounded from below, then \(F\) is a contraction if
(i) \(F(x) \leq F(y)\) for all \(x, y \in M\) and \(x \neq y\)
(ii) \(f(x) \leq f(y)\) for all \(x, y \in M\) and \(x \neq y\)

Throughout this paper, we assume that \(\kappa: [0, \infty) \to [0, \infty)\) is an upper semi continuous function. Now prove our main results.

2. Main Results

Definition 2.1
Suppose \((X,d)\) be a metric space endowed with a directed graph \(G\). Let us consider the mappings \(f, g: X \to X\) with defining the sets
(i) \(\{f(x), y \in X \times X: (f(x), y, S(\gamma, y)) \in \mathcal{E}(G), (f(x), y, S(\gamma, y)) \in \mathcal{E}(G^-)\}\) and (i) \(f, g\) is edge preserving. i.e., \((f(x), y, S(\gamma, y)) \in \mathcal{E}(G), (f(x), y, S(\gamma, y)) \in \mathcal{E}(G^-)\)
(ii) \(S, T\) is edge preserving. i.e., \((f(x), y, S(\gamma, y)) \in \mathcal{E}(G), (f(x), y, S(\gamma, y)) \in \mathcal{E}(G^-)\)
implies \((T(x), y, S(\gamma, y)) \in \mathcal{E}(G), (T(x), y, S(\gamma, y)) \in \mathcal{E}(G^-)\)
(iii) for all \(x, y, z \in X\) and for
\[
\begin{align*}
(f(x), y, (S(x), y, T(x), z)) \in \mathcal{E}(G) \\
(f(y), (S(x), y, T(x), z)) \in \mathcal{E}(G^-)
\end{align*}
\]
define
\[
d(S(x, y, T(x), z)) \leq \max \left\{ k(\psi(f(x), y)), k(\psi(y, x, T(x), z)) \right\}
\]
\[
\begin{align*}
[\psi(f(x), y) - \psi(y, x, T(x), z)] & - \max \left\{ k(\psi(f(x), y)), k(\psi(y, x, T(x), z)) \right\} \\
&= \psi(f(x), y) - \psi(y, x, T(x), z)
\end{align*}
\]
where \(\psi, \phi: X \times X \to [0, \infty)\) are lower semi continuous functions.

Theorem 2.2:
Let \(S, T: X \times X \to X\) and \(f, g: X \to X\). Suppose that \(S, T, f, g\) are \(G\)-edge preserving and satisfies \(G - f\) contraction. Let \(S(X \times X) \subseteq \mathcal{F}(X)\) and \(T(X \times X) \subseteq \mathcal{F}(X)\). Also let \(\{x\}, \{y\}, \{z\}, \{w\}\) be sequences in the metric space \((X, d)\) endowed with a directed graph \(G\). Then the following statements are true.
(i) \((f, g) \in \mathcal{E}(G)\) and \((f, g) \in \mathcal{E}(G^-)\) implies \((S(x, y, T(x), z)) \in \mathcal{E}(G)\)
and \((S(x, y, T(x), z)) \in \mathcal{E}(G^-)\) for all \(x, y, z \in X\) and \(f, g\) are \(G\)-edge preserving. i.e., \((S(x, y, T(x), z)) \in \mathcal{E}(G), (S(x, y, T(x), z)) \in \mathcal{E}(G^-)\)
(iii) \(\{x\}\) and \(\{y\}\) are cauchy sequences and there exists \(x', y' \in X\) such that \(x_n \to x'\) and \(y_n \to y'\).

Proof:
We have \(S(X \times X) \subseteq \mathcal{F}(X)\) and \(T(X \times X) \subseteq \mathcal{F}(X)\) so let us define the following sequences
\[
\begin{align*}
\Omega_2n &= \{x_{2n+1} \in \{x\}, y_{2n+1} \in \{y\}\} \\
\eta_2n &= \{x_{2n} \in \{x\}, y_{2n} \in \{y\}\}
\end{align*}
\]
where \(\{x\}, \{y\}\) are sequences in \(X\) such that \(x_{2n} \to x'\) and \(y_{2n} \to y'\).
\[\leq \max \left\{ \kappa(\psi(\eta_{2n-1}, \eta_{2n})) \right\} \left\{ \psi(\eta_{2n-1}, \eta_{2n}) - \psi(\eta_{2n}, \eta_{2n+1}) \right\}.\]

Since \(\Omega_{\infty} = \Omega_{\infty+1}, \Omega_{2n} = \eta_{2n+1}\) and \(\Omega_{2n+1} = \eta_{2n+1}\), for \(n = 1, 2, 3, \ldots\) so from Lemma 1.8 we have \(\{\psi(\Omega_{2n}, \Omega_{2n+1})\}\) and \(\{\psi(\Omega_{2n+1}, \Omega_{2n+2})\}\) are non-increasing.

Let \(\lim_{n \to \infty} \psi(\Omega_{2n}, \Omega_{2n+1}) = \lambda_1 \lim_{n \to \infty} \psi(\eta_{2n}, \eta_{2n+1}) = \mu\) for some \(\lambda_1, \mu \geq 0\).

Since \(\kappa\) is upper semi continuous function so we have \(\lim_{n \to \infty} \sup(\psi(\eta_{2n}, \eta_{2n+1})) = \kappa(\mu)\).

So for any \(m \in \mathbb{N}\) with \(n \geq n_0\) we have \(\lim_{n \to \infty} \kappa(\psi(\Omega_{2n}, \Omega_{2n+1})) = \kappa(\lambda) + 1\) and \(\lim_{n \to \infty} \kappa(\psi(\eta_{2n}, \eta_{2n+1})) = \kappa(\mu) + 1\). Therefore

\[d(\Omega_{2n}, \Omega_{2n+1}) \leq (\kappa(\lambda) + 1)(\psi(\Omega_{2n-1}, \Omega_{2n}) - \psi(\Omega_{2n}, \Omega_{2n+1}))\]

For \(m > n\), we have

\[d(\Omega_{2n}, \Omega_{2n+1}) = \sum_{i=n+1}^{m} d(\Omega_{2i-1}, \Omega_{2i}) \leq (\kappa(\lambda) + 1)(\psi(\Omega_{2n-1}, \Omega_{2n}) - \psi(\Omega_{2n}, \Omega_{2n+1})).\]

Similarly we can prove that \(\{\eta_{2n}\}\) is a Cauchy sequence.

Since \((X, d)\) is a complete metric space, there exists \(x^* \in X\) such that \(\lim_{n \to \infty} \Omega_{2n} = x^*\).

**Theorem 2.3:**

In addition to Theorem 2.2, assume that \(f, g\) are continuous and (i) \(f\) commutes with \(S\) and \(g\) commutes with \(T\) or (ii) \((X, d, G)\) has the property (A).

Then \(\text{CCon}(\text{C}) \neq \emptyset\) iff \((X \times X)_{\text{fg}} \neq \emptyset\).

**Proof:**

Case (i): Let \(f\) commutes with \(S\) and \(g\) commutes with \(T\).

Suppose \(\text{CCon}(\text{C}) \neq \emptyset\).

Then there exists \((\xi, \tau) \in \text{CCon}(\text{C}) \neq \emptyset\).

So \((R, S(\xi, \tau) = (R, S(\xi, \tau) = (R, T(\tau, \xi)).\)

Case (ii): Let \(S\) be an \(\infty\)-continuous function such that \(\lim_{n \to \infty} f(S(\eta_{2n}, \eta_{2n+1})) = f(x^*)\).

Similarly we can prove that \(g^* = g(x^*, y^*)\).

**Theorem 2.4:**

Suppose that hypothesis of Theorem 2.3 holds. Besides, let for every \((a, b^*), (c^*, d^*) \in (X \times X),\) there exists \((\xi, \tau) \in (X \times X)\)

such that

\(S(a, b^*), T(\xi, \tau) \in E(G), (S^*(b^*, a), T(\tau, \xi) \in E(G^-)\)

Then \(S(\xi, \tau), T(\tau, \xi) \in E(G), (S^*(\xi, \tau), T(\tau, \xi) \in E(G^-)\)

Also

\(S(\xi, \tau), T(\tau, \xi) \in E(G), (S(\xi, \tau), T(\tau, \xi) \in E(G^-)\)

Then \(S, T, f, g\) have a unique CCFP.

**Proof:**

Let \((a, b^*), (c^*, d^*)\) be \(C\) points of \(S, T, f, g\).

Then

\(fa = f(a, b^*), fb = f(b^*, a), ga = g(a, b^*), gb = g(b^*, a)^+ fc = f(c^*, d^*), fd^+ = f(d^*, c^*), gc = g(c^*, d^*), gd^+ = g(d^*, c^*)\)

Set \(T(\xi, \tau) = (\xi, \tau) \in E(G), (S^*(\xi, \tau), T(\tau, \xi) \in E(G^-)\)

Then \(S(\xi, \tau), T(\tau, \xi) \in E(G), (S(\xi, \tau), T(\tau, \xi) \in E(G^-)\)

This shows that \(S, T, f, g\) have a unique CCFP.
Now by using the edge preserving property repeatedly for \( n \geq 1 \), we have
\[
(f^a, g^{2n+2}) \in E(G), \quad (f^b, g^{2n+2}) \in E(G^{-1}) \text{ and } \quad (f^c, g^{2n+2}) \in E(G).
\]
Now
\[
d(d(f^a, f^c)) \leq d(f^a, g^{2n+1}) + d(g^{2n+1}, f^c) = d(S(f^a, b^c), T(g^{2n+1}, g^{2n+1})) + d(T(g^{2n+1}, g^{2n+1}), S(c^e, d^e)) = d(S(f^a, b^c), T(g^{2n+1}, g^{2n+1})) + d(T(g^{2n+1}, g^{2n+1}), S(c^e, d^e)) \leq \max(k(\psi(f^a, g^{2n+1})), k(\psi(f^a, g^{2n+1}))) \leq 0.
\]
Therefore \( f^a = f^c \) similarly, we get \( f^b = f^d \), \( g^a = g^c \) and \( g^b = g^d \).

Now for an arbitrary \((m, n) \in X \times X\),
\[
\|f^a - f^c\| = \|f^b - f^d\| = \|g^a - g^c\| = \|g^b - g^d\|.
\]
Therefore \( f^a = f^c \) and \( g^a = g^c \).

Now consider \( \|f^b - f^d\| < \|f^a - f^c\| \) contractive,
\[
\|f^b - f^d\| < \|f^a - f^c\| \Rightarrow \lim_{n \to \infty} f^b = f^d \quad \text{and} \quad \lim_{n \to \infty} g^b = g^d.
\]
Therefore \( f \) and \( g \) are CCFP.

This shows that \((m, n)\) is a CCFP of \( f \) and \( g \).

Finally to prove the uniqueness of \((m, n)\),
\[
\|f^m - f^n\| = \|g^m - g^n\|.
\]
Hence the theorem is proved.

It is easy to prove our main results for two maps.

**Corollary 1:**
Suppose \((X, d)\) to be a complete metric space endowed with a directed graph \( G \). Let us consider the mappings \( T : X \times X \to X \) and \( f : X \to X \) with the following sets
\[
(X \times X)_{tr} = \{(x, y) \in X \times X : (f(x), T(x, y)) \in E(G), (f(y), T(y, x)) \in E(G^{-1})\}
\]
Then Theorem 2.2, 2.3, 2.3 holds for the two maps \( T, f \) with the \( G - f_{1}\)-contaction
\[
d(T(x, y), T(x, z)) = \max(k(\psi(f(x, y), z), k(\psi(T(x, y), T(x, z)))) + \min(k(\psi(f(x, y), z), k(\psi(T(x, y), T(x, z))))).
\]
where \( \psi, \phi : X \times X \to [0, \infty) \) are lower semi continuous functions.

**3. Application to Integral Equations**

In this section, to discuss the application of our main results we establish an existence theorem in a metric space with graph for the solution of the integral equations.

Let us consider the following integral equations:
\[
\int_{0}^{t} f(t, \chi(s), \gamma(s))ds, \quad \gamma(t) = \int_{0}^{t} f(t, \gamma(s), \chi(s))ds.
\]
for all \( t \in [0, 1] \) and \( f \) \( X \times R \to R \). Let \( X = C(I, R) \) and define \( d : X \times X \to R \) as \( d(x, y) = |x - y| \) for \( x, y \in X \).

Define \( \psi : X \times X \to [0, \infty) \) for \( \psi(a, b) = |a - b| \), and \( \phi : X \times X \to [0, \infty) \) for \( \phi(a, b) = \frac{|a-b|}{2} \) and define \( \kappa : [0, +\infty) \to [0, +\infty) \) as
\[
\kappa(t) = \begin{cases} \frac{3}{2} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}
\]
Further define graph \( G \) on \( X \) using partial ordering relation.

i.e., \( x \in X \), \( y \in X \), \( x \leq y \text{ if and only if } (x, y) \in E(G) \).

\( E(G^{-1}) = \{(u, v) : (v, u) \in E(G)\} \)

This shows that the solution of 3.1 is a coupled coincidence point of the mapping \( T, f \), provided we verify the conditions of Corollary 1.

Since graph \( G \) is defined on \( X \) and \( g : X \to X \) is edge preserving.

Now using edge preserving property, suppose that \( r, t, u, v \in R \) such that \( x, y, z, \in X \).

Then there exists at least one solution of 3.1 in \( X = C(I, R) \).

**Theorem 3.1:**

**Proof:**
Define \( T : X \times X \to X, g : X \to X \) by
\[
T(x, y) = \int_{0}^{t} f(t, \chi(s), \gamma(s))ds, \quad g(x) = \int_{0}^{t} f(t, \gamma(s), \chi(s))ds.
\]
Then 3.1 is equivalent to \( g(x) = T(x, y) \).

This shows that the solution of 3.1 is a coupled coincidence point of the mapping \( T, g \), provided we verify the conditions of Corollary 1.

Hence the theorem is proved.
Therefore L.H.S \leq \text{R.H.S}

Now, condition (iii) of hypothesis implies that there exists \((x_0, y_0) \in X \times X\) such that

\[ x_0(t) \leq \int_0^t f(t, x_0(s), y_0(s))ds \quad \text{and} \quad y_0(t) = g(y_0(t)) \quad \text{so we have} \]

\[ g(x_0) \leq T(x_0, y_0) \quad \text{and} \quad T(y_0, x_0) \leq g(y_0) \]

Hence by the definition of \(g\) we have, \(x^* = g(x^*) = T(x^*, y^*)\) and \(y^* = g(y^*) = T(y^*, x^*)\).

Therefore \((x^*, y^*)\) is a solution of the equation 3.1.

4. Conclusion

In this paper we defined two different graphs sets with some properties and intersection of those two graphs as new set and finally proved that all the four maps in the graph have a unique common coupled fixed point.

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References


